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SUPERCRITICAL FLOW GENERATING A SOLITARY-LIKE WAVE ABOVE A BUMP*

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Abstract. We concern with a steady free surface flow in a channel having a bump on the bottom. Far upstream the flow is uniform, and it is disturbed by the bump; so that it generates waves on the fluid surface. In studying the problem, we derive the equation of the surface elevation as a forced Korteweg de Vries (fKdV) equation, with the forcing term representing the bump on the channel. Based on the potential function, the equation is obtained by a series expansion of the small parameter, defined as the ratio between the fluid depth and the horizontal length. In case the bottom of the cannel is a bump of a secant-hyperbolic function, we can solve the fKdV equation analytically in a solitary-like wave which is symmetric to the vertical axis. For a semicircular bump, the solution can be determined by a shooting method, until a certain width of the bump, but only for supercritical flow.

Key words. Free surface flow, forced Korteweg de Vries equation, series expansion, solitary wave.

1. Introduction. A 2-D flow is considered over a bump on the bottom of a channel. We assume that the fluid is ideal and the flow is irrotational, so that the flow can be described in term of a potential function. Far upstream the flow is uniform with velocity u_0 and depth H. The effect of the bump is to generate waves traveling upstream and downstream from the bump. Wiryanto [9] examined the unsteady problem, formulated as Boussinesq type equations, and solved numerically. His results indicate that the steady wave is set up when the flow upstream is supercritical, i.e. the Froude number $F = u_0/\sqrt{gH}$ greater than 1. Here g is the acceleration of gravity. The solution behaves like a solitary wave.

Some works related to supercritical flows can be seen in Forbes and Schwartz [5], Vanden-Broeck [8] and Forbes [4], who solved numerically the steady problem, direct from the exact equations, without approximating the governing equations into a single equation, for a semicircular bump. Their numerical computations indicated that there were two branches of solutions for F greater than a certain number, namely F_+, and no solution exists below F_+. The size of the bump corresponds to the size of the wave, which approaches to uniform state or a solitary wave by decreasing the size of the bump. For subcritical flows, $F < 1$, steady free surface with a train of waves is the typical of the solution. The solution exists for the Froude number less than a certain number F_-, i.e. $F < F_- < 1$. Wiryanto [10] obtained a train of waves behind an obstruction. Zhang and Zhu [11] solved numerically the free surface flow past a bottom obstruction, producing waves. The wave resistance calculated and was compared to the predicted result from the linear model. However, flows in the transition region $F_- < F < F_+$ can possible produce waves, but for unsteady case, such as obtained in Wiryanto [9].

Based on the results in the above references, we are interested in observing the steady solitary-like wave generated by supercritical flow. The governing equation is simplified into a single equation of the surface elevation, that we call forced Korteweg de Vries (fKdV) equation, containing one non-dimensional parameter Froude number F, defined based on the uniform upstream. The solution of the equation is determined to be compared to the numerical solutions from the exact equation.

In deriving the fKdV equation, we formulate the equation from the disturbance of the potential function $\phi = u_0 x$, for horizontal axis x. This stage is required to separate the uniform flow from the potential function, that cannot be seen directly in Euler equations such as in the derivation by Shen, et. al. [7], but it was obtained for the first order of their series expansion which physical interpretation. Similarly, for two fluid system, Shen [6] derived the model having a fKdV type equation, both for interfacial wave and the surface wave.

In solving the fKdV equation, we first consider the forcing term in form of a secant-hyperbolic bump, corresponding to the solution of the KdV equation. One solution of secant-hyperbolic type can be obtained only for a certain Froude number that depends on the height and width of the bump. This is also presented in Chardard, et. al. [3], Camassa and Wu [1], and Camassa and Wu [2]. The second case is the forcing term representing a bump in a finite interval, but it is symmetric in the space, such as a semicircular. A numerical approach is used to solve the problem. A shooting method is able to solve the problem for a symmetric free-surface profile. As the result, for given a bump, a supercritical flow can

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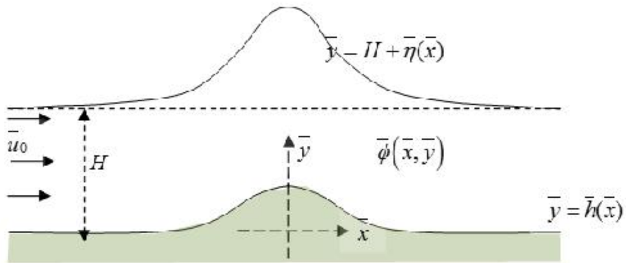

FIG. 2.1. *Sketch of the flow and coordinates.*

produce two solitary-like surface waves having different amplitude, and there is a minimum quantity of the flow, expressed as Froude number, where the solitary solution exists. The effect of the bump width is also observed.

2. Problem Formulation. The configuration of the flow over a bump is shown in Figure 1. As the system of coordinates, we choose Cartesian with the horizontal $\bar{x}$-axis along the flat bottom channel, and the vertical $\bar{y}$-axis perpendicular to the horizontal axis. The bump is described by $\bar{y} = \bar{h}(x)$; we only consider symmetric bump, $\bar{h}(-\bar{x}) = \bar{h}(\bar{x})$, with respect to the vertical axis. The surface elevation is $\bar{y} = H + \bar{\eta}(\bar{x})$, with uniform depth H as $|\bar{x}| \to \infty$, and the potential function is denoted by $\bar{\phi}$.

The problem of determining the steady state solution for the flow is to solve

$$\bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{y}\bar{y}} = 0 \tag{2.1}$$

in the flow domain $-\infty < \bar{x} < \infty$, $\bar{h}(\bar{x}) < \bar{y} < H + \bar{\eta}(\bar{x})$; with conditions

$$\bar{\phi}_{\bar{y}} - \bar{\phi}_{\bar{x}}\bar{\eta}_{\bar{x}} = 0 \tag{2.2}$$

$$\frac{1}{2}\left(\bar{\phi}_{\bar{x}}^2 + \bar{\phi}_{\bar{y}}^2\right) + g(H + \bar{\eta}) = \frac{1}{2}\bar{u}_0^2 + gH \tag{2.3}$$

along the free surface $\bar{y} = H + \bar{\eta}(\bar{x})$, and

$$\bar{\phi}_{\bar{y}} - \bar{\phi}_{\bar{x}}\bar{h}_{\bar{x}} = 0 \tag{2.4}$$

along the bottom $\bar{y} = \bar{h}(\bar{x})$. The right hand side of (2.3) represents the situation at the uniform stream.

We examine situations in which any horizontal length scale λ and the waves generated by the bump are very much larger than the depth; this enables us to define a small parameter $\varepsilon = (H/\lambda)^2$. We also assume that the amplitude of the bump is small, so that we can express the stream as the uniform flow $\bar{u}_0\bar{x}$ disturbed by smaller term, namely $\bar{\Phi}$. Hence, the potential function becomes $\bar{\phi} = \bar{u}_0\bar{x} + \bar{\Phi}$.

In observing the effect of the bottom topography, we first non-dimensionalize the variables with respect to the uniform depth H and the speed $\sqrt{gH}$. We then scale the variables by involving ε, so that we can analyze the contribution of the variables in each order. Both steps are expressed by introducing

$$(x, y) = \left(\varepsilon^{\frac{1}{2}}\bar{x}, \bar{y}\right)/H, \quad \Phi = \varepsilon^{\frac{1}{2}}\bar{\Phi}/\left(H\sqrt{gH}\right),$$

$$\eta = \bar{\eta}/H, \quad h = \varepsilon^{-2}\bar{h}/H.$$

The equation (2.1)-(2.4) becomes, presented in Φ,

$$\varepsilon\Phi_{xx} + \Phi_{yy} = 0 \tag{2.5}$$

subject to the conditions at the free surface

$$\Phi_y - \varepsilon\eta_x(F + \Phi_x) = 0 \tag{2.6}$$

$$F\Phi_x + \frac{1}{2}\Phi_x^2 + \frac{1}{2\varepsilon}\Phi_y^2 + \eta = 0 \tag{2.7}$$

and the condition at the bottom

$$\Phi_y - \varepsilon^3 h_x \left(F + \Phi_x\right) = 0 \tag{2.8}$$

where $F = \bar{u}_0/\sqrt{gH}$, the Froude number of the flow. The behavior of the flow is strongly dependent on the Froude number with critically occurring for $F \approx 1$. With this in mind we write $F = F^{(0)} + \varepsilon F^{(1)} + \varepsilon F^{(2)}$, and we expand Φ and η in an asymptotic series as

$$\Phi = \varepsilon \Phi^{(1)} + \varepsilon^2 \Phi^{(2)} + \cdots \tag{2.9}$$

$$\eta = \varepsilon \eta^{(1)} + \varepsilon^2 \eta^{(2)} + \cdots \tag{2.10}$$

$F^{(0)}$ is the critical Froude number, and has to be determined. We then substitute (2.9) and (2.10) in (2.5)-(2.8), yielding a sequence of equations and the boundary conditions for the successive approximations $\Phi^{(n)}$.

The first three order are as follow. The equation for the first order is

$$\Phi^{(1)}_{yy} = 0 \tag{2.11}$$

subject to

$$\Phi^{(1)}_y (x,1) = 0, \tag{2.12}$$

$$F^{(0)} \Phi^{(1)}_x (x,1) + \eta^{(1)} = 0 \tag{2.13}$$

and

$$\Phi^{(1)}_y (x,0) = 0. \tag{2.14}$$

The equation for the second order is

$$\Phi^{(1)}_{xx} + \Phi^{(2)}_{yy} = 0 \tag{2.15}$$

subject to

$$\Phi^{(2)}_y (x,1) - F^{(0)} \eta^{(1)}_x = 0 \tag{2.16}$$

$$F^{(1)} \Phi^{(2)}_x (x,1) + F^{(1)} \Phi^{(1)}_x (x,1) + \left(\Phi^{(1)}_x (x,1)\right)^2 + \eta^{(2)} = 0 \tag{2.17}$$

and

$$\Phi^{(2)}_y (x,0) = 0. \tag{2.18}$$

The equation for the third order is

$$\Phi^{(2)}_{xx} + \Phi^{(3)}_{yy} = 0 \tag{2.19}$$

subject to

$$\Phi^{(3)}_y (x,1) - \eta^{(2)}_x - F^{(1)} \eta^{(1)}_x - \Phi^{(1)}_x (x,1)\, \eta^{(1)}_x = 0 \tag{2.20}$$

$$\begin{aligned} &\Phi^{(3)}_x (x,1) + F^{(1)} \Phi^{(2)}_x (x,1) + F^{(2)} \Phi^{(1)}_x (x,1) + \\ &\frac{1}{2}\left(\Phi^{(2)}_y (x,1)\right)^2 + \Phi^{(1)}_x (x,1)\, \Phi^{(2)}_x (x,1) + \eta^{(3)} = 0 \end{aligned} \tag{2.21}$$

and

$$\Phi^{(3)}_y (x,0) - h_x = 0. \tag{2.22}$$

From the first order, $\Phi^{(1)}$ is quadratic in y, and the boundary condition (2.12) and (2.14) on $\Phi_y^{(1)}$, then give

$$\Phi^{(1)}(x,y) = \Phi^{(1)}(x) \tag{2.23}$$

a function of x alone. The boundary condition (2.13) then gives the consistency condition

$$\Phi_x^{(1)}(x) = -\frac{\eta^{(1)}(x)}{F^{(0)}} \tag{2.24}$$

higher order terms are required to explicitly determine $\Phi^{(1)}$, $\eta^{(1)}$.

Now, we work on the second order. We integrate (2.15) with respect to y, and condition (2.18) is used, so that we have

$$\Phi_y^{(2)}(x,y) = \frac{\eta_x^{(1)}}{F^{(0)}} y. \tag{2.25}$$

The value $F^{(0)}$ can be obtained by substituting (2.25) to the boundary condition (2.16), so that we have $F^{(0)} = 1$. The result (2.25) is then integrated, and the boundary condition (2.17) is used to determine the constant of integration. Therefore we have

$$\Phi_x^{(2)}(x,y) = \frac{1}{2}\eta_{xx}^{(1)}\left(y^2-1\right) + F^{(1)}\eta^{(1)} - \frac{1}{2}\left(\eta^{(1)}\right)^2 - \eta^{(2)}. \tag{2.26}$$

We obtain so far the relation $\Phi^{(1)}$, $\Phi^{(2)}$, $\Phi^{(3)}$ to $\eta^{(1)}$. In the next order we expect $\Phi^{(1)}$, $\Phi^{(2)}$, $\Phi^{(3)}$ can be eliminated, so that we have an equation of $\eta^{(1)}$, without appearing $\eta^{(2)}$.

We substitute (2.26) in (2.19), and it is integrated with respect to y. The constant of integration can be obtained by using the boundary condition (2.22), so that we have

$$\Phi_y^{(3)}(x,y) = -\frac{1}{2}\eta_{xxx}^{(1)}\left(\frac{1}{3}y^3 - y\right) - F^{(1)}\eta_x^{(1)}y + \eta^{(1)}\eta_x^{(1)}y + \eta_x^{(2)}y + h_x. \tag{2.27}$$

The equation of $\eta^{(1)}$ is obtained by substituting (2.27) into (2.20), i.e.

$$\frac{1}{3}\eta_{xxx}^{(1)} - 2F^{(1)}\eta_x^{(1)} + 2\eta^{(1)}\eta_x^{(1)} + h_x = 0 \tag{2.28}$$

In case $h_x = 0$, equation (2.28) is well known as the steady KdV equation, and has analytical solution in term of secant-hyperbolic function. On the other hand, (2.28) is then called forced KdV, with forcing term $h_x \neq 0$.

3. Solitary Wave Solution . In solving (2.28), we consider two cases of the bottom topography $h(x)$, i.e. the disturbance of the flow is in a large and short interval. We present each case in different subsection, solving (2.28) analytically and numerically.

3.1. Analytical Solution . Before we solve (2.28), we review in determining the secant-hyperbolic solution of $\eta^{(1)}$ for $h_x = 0$. This type of function is possible to use it as the disturbance of the flow $h(x)$, and we expect it gives similar solution.

The homogeneous equation of (2.28) is integrated with respect to x, and use the condition η and its derivation tend to 0 as $|x| \to \infty$. We obtain

$$\frac{1}{3}\eta_{xx}^{(1)} - 2F^{(1)}\eta^{(1)} + \left(\eta^{(1)}\right)^2 = 0.$$

This equation is then multiplied by $\eta_x^{(1)}$ and integrated, giving

$$\left(\eta_x^{(1)}\right)^2 - 6F^{(1)}\left(\eta^{(1)}\right)^2 + 2\left(\eta^{(1)}\right)^3 = 0$$

The non zero solution of this equation is

$$\eta^{(1)}(x) = 3F^{(1)}\text{sech}^2\left(x\sqrt{\frac{3}{2}F^{(1)}}\right).$$

In that form we choose the position of the maximum value of $h^{(1)}$ at $x = 0$, otherwise we should shift x to certain value, namely x_0.

Based on the homogeneous solution, we determine $h^{(1)}$ satisfying (2.28) for the forcing term

$$h(x) = G\operatorname{sech}^2(bx)$$

with constant G and b, for $-\infty < x < \infty$. We suppose the solution is

$$\eta^{(1)}(x) = a\operatorname{sech}^2(bx).$$

It is possible the solution contain a term of $\operatorname{sech}^4(bx)$, but we only consider the first type solution.

The relations among a, b and G are determined, by substituting $\eta^{(1)}$ and h into (2.28), and we collect the coefficients of $\operatorname{sech}^2(bx)$ and $\operatorname{sech}^4(bx)$ separately, we have

$$\left(\frac{4}{3}ab^2 - 2aF^{(1)}\right)\operatorname{sech}^2(bx) - \left(2ab^2 - a^2\right)\operatorname{sech}^4(bx) = -G\operatorname{sech}^2(bx).$$

Therefore, we obtain

$$2b^2 - a = 0, \qquad \frac{4}{3}ab^2 - 2aF^{(1)} + G = 0.$$

We can describe these relation as follows. When a solitary bump with height G and width b is used to disturb a flow, a solitary wave can be obtained only for Froude number

$$F^{(1)} = \frac{2}{3}b^2 + \frac{G}{4b^2}.$$

The solitary wave is of height $a = 2b^2$. A unique solution of (2.28) is obtained for this case, which is different with the numerical solution obtained in [5].

3.2. Numerical Solution . In this stage, we solve (2.28) for the case where the flow is disturbed in a certain interval. A bump is a cosine function, placed in $x \in (-L, L)$, namely

$$h(x) = \begin{cases} 0, & |x| > L \\ G\cos(\pi x/(2L)). & |x| < L \end{cases} \tag{3.1}$$

The bump is semicircular with width $2L$. The difficulty in solving (2.28) analytically is the equation (2.28) contain nonlinear term. So, we propose to solve (2.28) numerically. But, we assumed that the solution be symmetric to the vertical axis, and satisfies $\eta^{(1)}(0) = 0$, so that it is enough to determine $\eta^{(1)}$ for $x < 0$.

In designing the numerical procedure, we first consider the solution of (2.28) for $x < -L$. Referring to the homogeneous solution of (2.28), we have

$$\eta^{(1)}(x) = 3F^{(1)}\operatorname{sech}^2\left(\sqrt{\frac{3}{2}F^{(1)}}\,(x - x_0)\right). \tag{3.2}$$

The constant x_0 is determined from the solution (2.28) in the domain $-L < x < 0$ satisfying the boundary conditions

$$\left.\frac{1}{3}\eta^{(1)}_{xx} - 2F^{(1)}\eta^{(1)} + \left(\eta^{(1)}\right)^2\right|_{x=-L} = 0 \tag{3.3}$$

and $\eta^{(1)}(0) = 0$. To solve that problem, we write it into a system of equations

$$\eta^{(1)}_x = s \tag{3.4}$$

$$s_x = 6F^{(1)}\eta^{(1)} - 3\left(\eta^{(1)}\right)^2 - 3G\cos(\pi x/(2L)) \tag{3.5}$$

with conditions $\eta^{(1)}(0) = A$, $s(0) = 0$. The wave height of $\eta^{(1)}$ is A, and it is determined by trial-error so that the condition (3.3) at $x = -L$ is satisfied. We solve (3.4) and (3.5) by Runge-Kutta method.

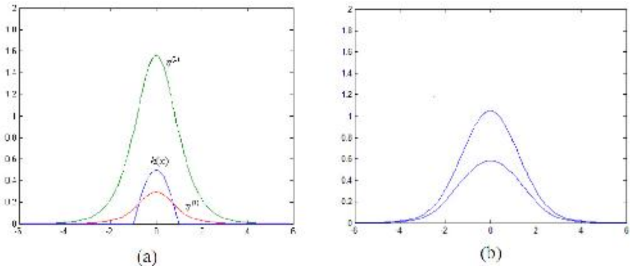

FIG. 3.1. *(a) The solution profile $\eta^{(1)}$ of (2.28), corresponding to $A = 1.56285$ and 0.2935 plotted together with the bottom topography $h(x)$ of (3.1) with $G = 0.5$, $F^{(1)} = 0.6$ (b) Two plots of $\eta^{(1)}$ for a bump $h(x)$ with $L = 2.5$.*

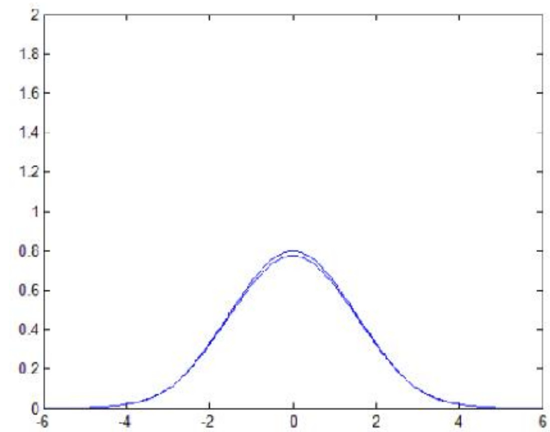

FIG. 3.2. *Plot of two solutions $\eta^{(1)}$ calculated using $G = 0.5$, $F^{(1)} = 0.6$ and $h(x)$ given (3.1) with $L = 2.98$. The wave heights are 0.775 and 0.798.*

When the correct value A is obtained, x_0 is determined from $\eta^{(1)}(-L)$, matched from the result of the Runge-Kutta method and (3.2).

In performing the numerical result, we present for the bump with $L = 1$. We calculate the solution for $G = 0.5$, $F^{(1)} = 0.6$. Our procedure gives two possibilities $A = 1.56285$ and 0.2935 to satisfy (3.3). The free surface corresponding to each A and the profile of the bottom topography are shown in Figure 2a. For different bump width, we perform the result for $L = 2.5$ in Figure 2b. The flow of $F^{(1)} = 0.6$ is disturbed with the same bump height $G = 0.5$. Our calculation gives two solutions with wave height $A = 1.0480$ and $A = 0.5830$. The difference of the wave height becomes smaller by increasing the bump width. This can be continued until both waves becoming one. A bump with $L = 2.98$ produces two solutions almost the same wave, with wave height $A = 0.775$ and 0.798, shown in Figure 3. A unique solution is obtained for $L = 2.985$, with wave height $A = 0.786$. Larger than that number, no solitary like wave can be obtained. This confirms to the analytical solution described in the previous subsection.

For other values of $F^{(1)}$, with disturbance (3.1) for $L = 1$, two solutions with different wave height are obtained. For smaller $F^{(1)}$, we obtain the difference between both values A becomes smaller, and there is minimum value for $F^{(1)}$. Smaller than that number, the solitary solution cannot be obtained. We also observe the effect of the bump height G. We collect the wave height A for various values $F^{(1)}$ and G. We then plot the data as shown in Figure 4. Two solutions are obtained for the same value . This agrees with the result in [5].

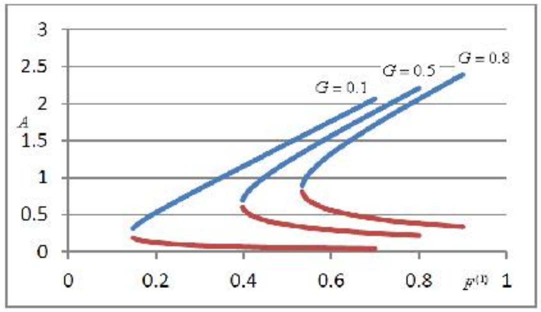

FIG. 3.3. *Plot of A versus $F^{(1)}$ for some values G, indicated near the curve. We calculated using $L = 1$.*

4. Conclusion. We have derived a forced Korteweg de Vries equation for free surface flow disturbed by a bump, from the governing equation based on the potential equation. The forcing term corresponds to the topography of the bottom channel. A unique analytical solution was obtained for the secant hyperbolic bump, and two solutions were obtained for bumps with relatively short width. Those two solutions became one for a certain bump width, and no solitary like solution for larger then that width.

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