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OPTIMAL CONTROL AND EQUILIBRIUM OF DELAYED S E I R MODEL WITH SATURATED INCIDENCE AND TIME DELAY IN CONTROL VARIABLES

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Abstract. In this paper we wil investigate the optimal control strategy of S E I R epidemic model with saturated incidence, which is use the mixed delay time in state and control variables. This strategy is related to vaccination program to minimize the number of the susceptible, exposed and also invected individuals and maximize the number of recovered individuals during the course of an epidemic. The result of our investigation is about existence and characterization of optimal solutions and their respective control of our model. We prove the existence of optimal solutions and optimal control. Then, to characterize this optimal control coresponding to optimal solutions, we analyze the Augmented Hamiltonian of our delayed control problem by Pontryagins Minimum Principle. The other goal, we will analyze the stability of the disease free and disease equilibrium of our delayed system.

Key words. SEIR, Augmented Hamiltonian, Pontryagins Minimum Principle

1. Introduction. In this paper we use S E I R (Susceptible Exposed Invected Recovered) as a basic model. Epidemic model with nonconstant population with logistic growth rule (in S I R model) has been analyzed by J.Z.Chang, et.al.,[3]. There are various type of incidence rate in epidemic system such us bilinear, nonlinear and saturated incidence.

In this paper, we use modified saturated incidence with two saturation factor $\frac{\beta SI}{\alpha_1 S+\alpha_2 I}$. that is measure the inhibitory effect of susceptible and also infected individuals. The saturated incidence is proposed by May and Anderson [7] in 1978. This incidence term is more realistic than the other incidence term in the sense of epidemic control. The modified saturated incidence is also use by Laarabi, et.al.[2]. The effect of each saturation factor, which is refer to α_1 and α_2, stems from natural epidemical control strategies. This strategies are based on taking appropriate preventive measures to the high level of the disease spread from susceptible and infected individuals respectively. In this paper we choose this modified saturated incidence to accomadate the protection measures of susceptible and infected individuals at a high infective.

One of method to control the spread of disease is vaccination strategy. The application of optimal control to the epidemic model has been analyzed by several author, such that [1, 2, 5, 6, 8]. Futher more, optimal control problem in epidemic model with time delay, in SIR model with bilinear incidence, has been analyzed by M.Elhia, et.al. [6]. In this paper, we try to find the existence of the optimal control of SEIR epidemic system with saturated incidence and nonconstant population with logistic growth rule. Time delay parameter also added to the state and control variable. Then, we also characterize this control to find this optimal value related to the optimal solutions.

This paper is organized as follows. In section 2, we will present our model formulation with its properties and assumption, futhermore, still in this section, we describe the optimal control problem which is related to optimal strategy of vaccination. Then, in section 3, we will analyze the existence of solutions and equilibria of our system. The existence of the optimal is presented in section 3. Finally, in section 4, we will use Pontryagins Minimum Principle to characterize the Augmented Hamiltonan of our control problem to obtain the optimal control.

2. Model Formulation. In this paper, we consider the SEIR with nonconstant population and generalized saturated incidence with discrete time delay τ_A. The time delay τ_A in this SEIR model is related to latent period of susceptible individuals become exposed. We take τ_A into bilinear term βSI, which is the part of the complete form of generalized saturated incidence term. We assume that the susceptible host population is assumed to have the logistic growth model with carrying capacity K and specific growth rate r, so the total host population is not constant. Hence, we have the generalized delayed SEIR model as follows :

$$\begin{aligned} \dot{S(t)} &= r\left(1-\frac{S(t)}{K}\right)S(t) - \frac{\beta S(t)I(t-\tau_A)}{1+\alpha_1 S(t)+\alpha_2 I(t)} \\ \dot{E(t)} &= \frac{\beta S(t)I(t-\tau_A)}{1+\alpha_1 S(t)+\alpha_2 I(t)} - \epsilon E(t) - \mu_1 E(t) \end{aligned} \tag{2.1}$$

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$$\dot{I(t)} = \epsilon E(t) - \gamma I(t) - \mu_2 I(t)$$
$$\dot{R(t)} = \gamma I(t) - \mu_3 R(t)$$

$$N(t) = S(t) + E(t) + I(t) + R(t) \tag{2.2}$$

where r, K, β, α_1, α_2, ϵ, μ_1, μ_2, μ_3 and γ are positive constants. α_1 and α_2 are paramaters that is measures the inhibitory effect of the susceptible and the infected individuals, respectively, ϵ is the rate at which exposed individuals become infectious μ_1,μ_2 and μ_3 represent the natural percapita death rate of exposed, infected and recovered individuals, γ is natural recovery rate. The intial values of system (2.1) $\varphi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ are defined in Banach Space :
$c_+ = \{\Phi \in C([-\tau, 0],\ R^4_+), : \Phi_1(\theta) = S(\theta), \Phi_2(\theta) = E(\theta), \Phi_3(\theta) = I(\theta), \Phi_4(\theta) = R(\theta), \theta \in [-\tau, 0]\}$ with $R^4_+ = \{(S, E, I, R) \in R^4, S \geq 0, E \geq 0, I \geq 0, R \geq 0\}$. By biological meaning, we assume that $\varphi_i \geq 0, i = 1, 2, 3, 4$.

Our main goal is to minimize the number of susceptible, exposed and also infected individuals and to maximize the number of recovered individuals by minimizing the respective objective functional into the model (2.1), we also include a control variable u, that is represents the percentage of susceptible individuals being vaccinated per unit time [6]. We also take the second discrete time delay τ_B into our system. It is measure the time taken by vaccinated susceptible individuals to move from the class of susceptible into recovered class. Thus, according to explanation by M.Elhia, et.al in [6]: at time τ_B only a percentage of susceptible individuals that have been vaccinated τ_B time unit ago, that is, at time $t - \tau_B$, are removed from the susceptible class and added to the recovered class. Hence, we get system (2.1) with time delay in state and control variables as follows :

$$\begin{aligned} \dot{S(t)} &= r\left(1 - \frac{S(t)}{K}\right)S(t) - \frac{\beta S(t) I(t-\tau_A)}{1 + \alpha_1 S(t) + \alpha_2 I(t)} - u(t-\tau_B)s(t-\tau_B) \\ \dot{E(t)} &= \frac{\beta S(t) I(t-\tau_A)}{1 + \alpha_1 S(t) + \alpha_2 I(t)} - \epsilon E(t) - \mu_1 E(t) \\ \dot{I(t)} &= \epsilon E(t) - \gamma I(t) - \mu_2 I(t) \\ \dot{R(t)} &= \gamma I(t) - \mu_3 R(t) + u(t-\tau_B)s(t-\tau_B) \end{aligned} \tag{2.3}$$

with $u(\theta) = 0$ is initial condition of u related to initial condition of system (2.1). The control u is assumed to be integrable in the sense of Lebesque integrable and also bounded with $0 \leq u \leq u_{max} < 1$, where u_{max} is a given constant.

3. The Existence of the Equilibrium. In this chapter we wiil find the equilibrium of system (2.3). When $\tau_A = 0$ and $\tau_B = 0$ system (2.3) is equivalent to

$$\begin{aligned} \dot{S(t)} &= r\left(1 - \frac{S(t)}{K}\right)S(t) - \frac{\beta S(t) I(t)}{1 + \alpha_1 S(t) + \alpha_2 I(t)} = 0 \\ \dot{E(t)} &= \frac{\beta S(t) I(t)}{1 + \alpha_1 S(t) + \alpha_2 I(t)} - \epsilon E(t) - \mu_1 E(t) = 0 \\ \dot{I(t)} &= \epsilon E(t) - \gamma I(t) - \mu_2 I(t) = 0 \\ \dot{R(t)} &= \gamma I(t) - \mu_3 R(t) = 0 \end{aligned} \tag{3.1}$$

From the above equations we get

$$\frac{\beta S I}{1 + \alpha_1 S + \alpha_2 I} = \frac{(\epsilon + \mu_1)(\gamma + \mu_2)}{\epsilon} I \tag{3.2}$$

Let define

$$P_1 = (\epsilon + \mu_1) \tag{3.3}$$
$$P_2 = (\gamma + \mu_2) \tag{3.4}$$

and then, from equation (3.2) we get

$$S^* = \frac{P_1P_2(1+\alpha_2 I^*)}{\beta\epsilon - P_1P_2\alpha_1} \tag{3.5}$$

If we subtitute the value of S^* in the equation (3.5) to the first equation of equation (3.1) we get

$$I^* = \frac{(P_5\alpha_2 + 2P_4 - \frac{P_3P_5}{\epsilon}) + \sqrt{(P_5\alpha_2 + 2P_4 - \frac{P_3P_5}{\epsilon})^2 + 4(P_5 - P_4)P_4\alpha_2^2}}{2P_4\alpha_2^2} \tag{3.6}$$

where

$$P_3 = \beta\epsilon - P_1P_2\alpha_1 \tag{3.7}$$

$$P_4 = \epsilon P_1 P_2 \tag{3.8}$$

$$P_5 = \epsilon K P_3 \tag{3.9}$$

If we define the threshold value

$$R_1 = \frac{1}{1 - 4(P_5 - P_4)P_4\alpha_2^2} \tag{3.10}$$

$$R_2 = \frac{1}{(P_5\alpha_2 + 2P_4 - \frac{P_3P_5}{\epsilon})} \tag{3.11}$$

If $R_1, R_2 \geq 0$ we get the positive value of I^* in equation (3.6).Hence we get the endemic equilibrium $E_1 = (S^*, E^*, I^*, R^*)$, with $E^* = \frac{\gamma+\mu_2}{\epsilon} I^*$ and $R^* = \frac{\gamma}{\mu_3}$. On the otherhand, if both R_1 and R_2 are equal to 1, we have two disease free equilibriums $E_0 = (0,0,0,0)$ and $E_K = (K,0,0,0)$.

4. The Optimal Control Problem and The Existence of Optimal Control. Our goal is to minimize the objective functional for fixed terminal time t_{end}

$$J(u) = \int_0^{t_{end}} (A_1S(t) + A_2E(t) + A_3I(t) - A_4R(t) + \frac{A_5}{2}u(t)^2)dt \tag{4.1}$$

where $A_i \geq 0$, for $i = 1,2,3,4,5$, is is weight that balance the size of terms. We use quadratic control term like the many author [2, 5, 6]. The control u is elements of admissible control U, which is defined by

$$U = \{0 \leq u \leq u_{maks} < 1, t \in [0, t_{end}]\} \tag{4.2}$$

where u is Lebesgue measurable. Obviously, we have the optimal control problem

$$min\{J(u) : u \in U\} \tag{4.3}$$

where $u \in U$. In order to find the existence of control optimal of system (2.3), we use same techniques which is used by Fleming and Rishel in [9] and also M. Elhia, et. al in [6].

Theorem 4.1. *Based on the control problem (2.3). There is an optimal control $u^* \in U$ such that $J(u^*) = min_{u\in U} J(u)$.*

Proof. By biological assumption its clear that the set of controls U, which is defined in (4.2) and the correspondig state variables is not empty. Then, the second step we will prove the control set U (4.2) is closed and convex. Based on definition of $u \in U$, which is for each$u \in U$ is bouded on the interval $[0, u_{maks}]$ and also convex on the interval. Based on the system (2.3), we can conclude that the right hand side of the state system is bounded by a linear function in the state and control variables using the boundedness of the solution. Its clear that the integrand in the form of objective functional (4.1) is convex on U. From the Lagrangian L of the objective functional (4.1) we have the positive constant c, k_1, k_2 such that satisfying □

$$L(S,E,I,R,u) \geq k_1 + k_2 \left|u^2\right|^{\frac{c}{2}} \tag{4.4}$$

5. The Characterization of The Optimal Control. From delayed control problem (2.3) we get the augmented Hamiltonian for delayed control problem

$$H = A_1S(t) + A_2E(t) + A_3I(t) - A_4R(t) + \frac{A_5}{2}u(t)^2 + \sum_{i=0}^{4}\lambda_i f_i \tag{5.1}$$

where f_i is the right hand side of the differential equation of the ith equation of the state variables. We will use Pontryagins maximum principle with delay in state and control variables [4] to get an optimal control u^* coresponding to optimal solutions (S^*, E^*, I^*, R^*). First, we will find the adjoints equations and transversality conditions of delayed control problem (2.3) that is satisfy the following conditions :

$$\dot{\lambda_1} = -\frac{\partial H(t)}{\partial S(t)} - \chi_{[0,t_{end}-\tau_B]}(t)\frac{\partial H(t+\tau_B)}{\partial S(t-\tau_B)}, \lambda_1(t_{end}) = 0 \tag{5.2}$$

$$\dot{\lambda_2} = -\frac{\partial H}{\partial E}, \lambda_2(t_{end}) = 0 \tag{5.3}$$

$$\dot{\lambda_3} = -\frac{\partial H(t)}{\partial I(t)} - \chi_{[0,t_{end}-\tau_A]}(t)\frac{\partial H(t+\tau_A)}{\partial I(t-\tau_A)}, \lambda_3(t_{end}) = 0 \tag{5.4}$$

$$\dot{\lambda_4} = -\frac{\partial H}{\partial R}, \lambda_4(t_{end}) = 0 \tag{5.5}$$

so we have the transversality conditions :

$$\lambda_1(t_{end}) = \lambda_2(t_{end}) = \lambda_3(t_{end}) = \lambda_4(t_{end}) = 0 \tag{5.6}$$

and from (5.2),(5.3),(5.4), and (5.5), we get

$$\begin{aligned}
\dot{\lambda_1} &= -A_1 + \lambda_1(r - \frac{2S^*}{K}) + \frac{(\lambda_1 - \lambda_2)\beta I^*(1+\alpha_2 I)}{(1+\alpha_1 S^* + \alpha_2 I^*)^2} + \chi_{[0,t_{end}-\tau_B]}(t)(\lambda_1(t+\tau_B) - \lambda_4(t+\tau_B))u^* \\
\dot{\lambda_2} &= -A_2 + (\lambda_2 - \lambda_3)\epsilon + \lambda_2\mu_1 \\
\dot{\lambda_3} &= -A_3 + \lambda_3(\gamma + \mu_2) - \mu_4\gamma + \frac{(\mu_2 - \mu_1)\alpha_2}{(1+\alpha_1 S^* + \alpha_2 I^*)^2} + \chi_{[0,t_{end}-\tau_A]}(t)\frac{(\lambda_1(t+\tau_A) - \lambda_2(t+\tau_A))\beta S^*}{(1+\alpha_1 S^* + \alpha_2 I^*)^2} \\
\dot{\lambda_4} &= A_4 + \lambda_4\mu_3
\end{aligned}$$

Futhermore, the optimal control u^* can be solved from the optimality condition :

$$\frac{\partial H(t)}{\partial u(t)} + \chi_{[0,t_{end}-\tau_B]}(t)\frac{\partial H(t+\tau_B)}{\partial u(t-\tau_B)}, \lambda_1(t_{end}) = 0 \tag{5.7}$$

That is

$$A_5u^* + \chi_{[0,t_{end}-\tau_B]}(t)(\lambda_4(t+\tau_B) - \lambda_1(t+\tau_B))S^* = 0 \tag{5.8}$$

By the bounds of u^* in admissible control set U, we get the optimal control u^* :

$$u^*(t) = min(u_{maks}, max(\frac{\chi_{[0,t_{end}-t]}(t)(\lambda_4(t+\tau_B)^+ - \lambda_1(t+\tau_B))S^*}{A_5})) \tag{5.9}$$

6. Conclusion. In this paper we get some result about properties and dynamics of control of our model. We prove the existence of our control optimal with Theorem (4.1). Finally, by using Pontryagins maximum principle to the Augmented Lagrangian of objective functional (4.1), which is contain delay time parameter, we obtain the explicit expression of optimal control of our optimal control problem.

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