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NUMERICAL SOLUTIONS TO UNSTEADY WAVE PROBLEMS OVER POROUS MEDIA

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Abstract. Unsteady wave problems over porous media are considered. Solutions to these problems are sought. We promote the use of an explicit predictor-explicit corrector method to solve the mathematical model over porous media. Because the numerical scheme is explicit, this numerical method is very easy to implement. Numerical experiments show the effectiveness of the method that we promote.

Key words. fluid flow, permeable bed, porous media, steady state solution, unsteady wave

1. Introduction. Fluid flow problems are challenging to solve, as they occur in various situations in closed and/or open channel. In closed channel, fluid flow can occur in pipes, tanks, etc. In open channel, fluid can flow in the athmosphere, ocean, river, etc.

This paper deals with fluid flow in open channel. Specifically we investigate unsteady waves generated by fluid flow over porous media. A mathematical model for this kind of flow was derived by Wiryanto [1]. This model has some relations to problems considered by Forbes and Schwartz [2], Vanden-Broeck [3], Forbes [4], and Shen et al. [5]. However those authors did not include porosity in their model. Flow over a bump without porosity was also discussed by Goutal and Maurel [6] and Mungkasi [7] using the shallow water equations.

Using potential function theory, Wiryanto [1] derived the mathematical model for unsteady waves generated by fluid flow over porous media. Note that we consider the model of Wiryanto [1] for the representation of our problem. After the derivation of the model, Wiryanto [1] showed some numerical solutions to the problem. The numerical method that he used was an explicit predictor-implicit corrector method. In his work all systems of linear equations were solved using Gauss-Seidel iteration. Nevertheless, solving this model using this implicit method is expensive computationally.

In the present paper we propose to solve the considered model using an explicit predictor-explicit corrector method. Because of its explicity, the method is very simple to implement. Moreover, its computational cost is cheaper than the implicit method.

The rest of this paper is organized as follows. The mathematical model and numerical methods are given in Section 2. Numerical results are presented in Section 3. We draw some concluding remarks in Section 4.

2. Governing equations and numerical methods. Unsteady waves generated by flow over porous media are governed by the following model of Wiryanto [1]

$$\frac{\partial \eta}{\partial t} + F\frac{\partial(\eta + u + h)}{\partial x} = A\frac{\partial^2 \eta}{\partial x^2}, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial(Fu + \frac{1}{F}\eta)}{\partial x} = 0, \tag{2.2}$$

which are two simultaneous partial differential equations. The model is represented in nondimensional quantities. This has balance law properties (see the work of Mungkasi [7] for balance laws). Here x is the space variable and t is the time variable. The horizontal reference axis is the fluid surface at rest. Furthermore, $\eta = \eta(x,t)$ describes the fluid surface, so it is the magnitude from the horizontal reference to the fluid surface. Positive value of η means that the water surface is above the horizontal reference axis. Variable $u = u(x,t)$ is the fluid velocity. The topography or fluid bed is given by $y = -(1 + h(x))$ and is measured from the horizontal reference to the fluid bed. The value of h is nonnegative. The constant

$$A = \frac{(d-1)R}{F} \tag{2.3}$$

corresponds to the absorbance parameters d and R of the porous bed. Here d is a positive constant such that $d > 1 + h(x)$ for all x and $y = -d$ is the vertical position where an impermeable bed lies beneath

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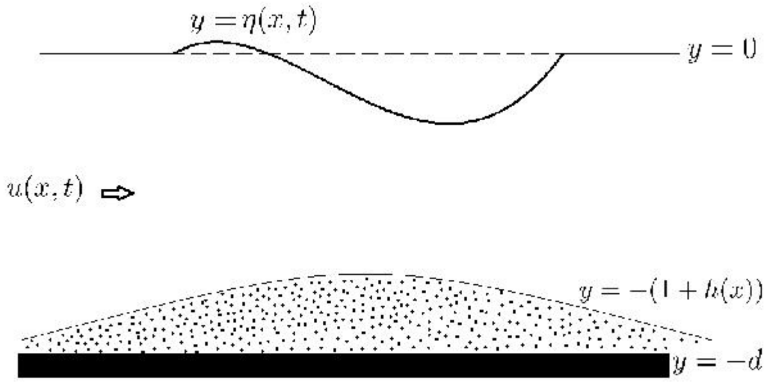

FIG. 2.1. *A sketch of the considered flow.*

the porous media. The constant R is the Reynolds number. In addition, F is the Froude number. Note that taking $A = 0$ means that the bed is without porousity and we call it perfectly impermeable.

An illustration of the considered flow is given in FIG. 2.1. Note that this figure is, without loss of generality, a dimensionless sketch. That is, the reference of the vertical height is set to be $h_0 = 1$, the reference of the horizontal length is $\lambda = 1$, and the acceleration due to gravity is assumed to be $g = 1$. Readers interested in the dimensional sketch of the flow can consult the work of Wiryanto [1].

We review Wiryanto's method [1] to solve this model. First the variables of space x and time t are discretized as $x_i = i\Delta x$ for $i = 0, 1, 2, \ldots$ and $t^n = n\Delta t$ for $n = 0, 1, 2, \ldots$. Then we approximate η and u pointwisely

$$\eta(x_i, t^n) \approx \eta_i^n \quad \text{and} \quad u(x_i, t^n) \approx u_i^n. \tag{2.4}$$

Here the subscript i represents the index of space discretization and the superscript n does the index of time discretization.

The implicit predictor-corrector method used by Wiryanto [1] is as follows. The predictor step is devoted to the prediction ${}^*\eta_i^{n+1}$ of the value η_i^{n+1}. It is evaluated using an explicit scheme applied to equation (2.1), that is,

$$\begin{aligned} {}^*\eta_i^{n+1} = \eta_i^n - \Delta t F \left[\frac{\eta_{i+1}^n - \eta_{i-1}^n}{2\Delta x} + \frac{u_{i+1}^n - u_{i-1}^n + h_x}{2\Delta x} \right] \\ + A \left[\frac{\eta_{i+1}^n - 2\eta_i^n + \eta_{i-1}^n}{\Delta x^2} \right]. \end{aligned} \tag{2.5}$$

The value ${}^*\eta_i^{n+1}$ is then used to calculate u_i^{n+1} with the discretization of equation (2.2) in the form of the implicit scheme

$$\begin{aligned} \frac{u_i^{n+1} - u_i^n}{\Delta t} + F \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1} + u_{i+1}^n - u_{i-1}^n}{4\Delta x} \\ + \frac{1}{F} \frac{{}^*\eta_{i+1}^{n+1} - {}^*\eta_{i-1}^{n+1} + \eta_{i+1}^n - \eta_{i-1}^n}{4\Delta x} = 0. \end{aligned} \tag{2.6}$$

Next the value of ${}^*\eta_i^{n+1}$ is corrected with the discretization of equation (2.1). That is in the form of the implicit scheme

$$\begin{aligned} \frac{\eta_i^{n+1} - \eta_i^n}{\Delta t} &+ F \frac{\eta_{i+1}^{n+1} - \eta_{i-1}^{n+1} + \eta_{i+1}^n - \eta_{i-1}^n}{4\Delta x} \\ &+ F \frac{u_{i+1}^{n+1} - u_{i-1}^{n+1} + u_{i+1}^n - u_{i-1}^n}{4\Delta x} + F h_x \\ &= A \left[\frac{\eta_{i+1}^{n+1} - 2\eta_i^{n+1} + \eta_{i-1}^{n+1} + \eta_{i+1}^n - 2\eta_i^n + \eta_{i-1}^n}{2\Delta x^2} \right]. \end{aligned} \tag{2.7}$$

Here h_x is the topography slope and is approximated using the second order central difference

$$h_x(x_i) \approx \frac{h_{i+1} - h_{i-1}}{2\Delta x}. \tag{2.8}$$

All resulting systems of linear equations were solved using Gauss-Seidel iteration. Note that this is a relatively expensive procedure computationally, because we have to solve large systems of linear equations.

Now we describe an alternative yet less expensive numerical procedure to solve the model (2.1) and (2.2). This model can be written as

$$\eta_t = G(\eta_x, u_x, h_x, \eta_{xx}), \tag{2.9}$$

$$u_t = H(u_x, \eta_x), \tag{2.10}$$

where G and H are given by

$$G = -F\eta_x - Fu_x - Fh_x + A\eta_{xx}, \tag{2.11}$$

$$H = -Fu_x - \frac{1}{F}\eta_x. \tag{2.12}$$

We propose to solve the model (2.1) and (2.2) using explicit predictor-explicit corrector method as follows. The predictor step is the third order explicit Adams-Bashforth scheme

$$^*\eta_i^{n+1} = \eta_i^n + \frac{\Delta t}{12}\left[23G_i^n - 16G_i^{n-1} + 5G_i^{n-2}\right], \tag{2.13}$$

$$^*u_i^{n+1} = u_i^n + \frac{\Delta t}{12}\left[23H_i^n - 16H_i^{n-1} + 5H_i^{n-2}\right]. \tag{2.14}$$

The values $^*\eta_i^{n+1}$ and $^*u_i^{n+1}$ are the predicted values of η_i^{n+1} and u_i^{n+1}, respectively. These predicted values are then used to compute $^*G_i^{n+1}$ and $^*H_i^{n+1}$.

Then the corrector step is the third order explicit Adams-Moulton scheme

$$\eta_i^{n+1} = \eta_i^n + \frac{\Delta t}{12}\left[5\,^*G_i^{n+1} + 8G_i^n - G_i^{n-1}\right], \tag{2.15}$$

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{12}\left[5\,^*H_i^{n+1} + 8H_i^n - H_i^{n-1}\right]. \tag{2.16}$$

This method was actually used by Wiryanto [8] in his other work to solve a different model, that is, a solitary-like wave generated by flow over a bump.

3. Numerical results. To test the performance of the promoted method, we consider a space domain $[0, 100]$. This space domain is discretized into finite difference nodes with $\Delta x = 0.05$. The time step is $0.001\Delta x$. This time step is chosen sufficiently small so that the method is stable. In the work of Forbes and Schwartz [2], Vanden-Broeck [3], Forbes [4], and Shen et al. [5], the fluid bed contains an obstruction in the positive vertical direction (convex from above). This present paper consider porous fluid bed and we shall also take an obstruction in the positive vertical direction. Care should be taken as our setting for fluid bed is given by $y = -(1 + h(x))$, whereas other authors might use the definition $y = 1 + h(x)$ for the fluid bed. Following Wiryanto [1], we set the variable $h(x)$ to be

$$h(x) = \begin{cases} -0.1\sin(0.1\pi(x-45)) & \text{if} \quad x \in [45, 55], \\ 0 & \text{if} \quad x \in [0, 45) \cup (55, 100]. \end{cases} \tag{3.1}$$

The boundary condition is taken as $\eta(0,t) = \eta(100,t) = 0$ and $u(0,t) = u(100,t) = 0$.

For all tests, we set $A = 0.3$. We investigate the propagation of waves generated by the flow over the defined bed. We take the Froude number $F = 0.2, 1.0, 1.6, 2.6$, as representations of subcritical ($F = 0.2$), critical ($F = 1.0$), and supercritical flows ($F = 1.6, 2.6$).

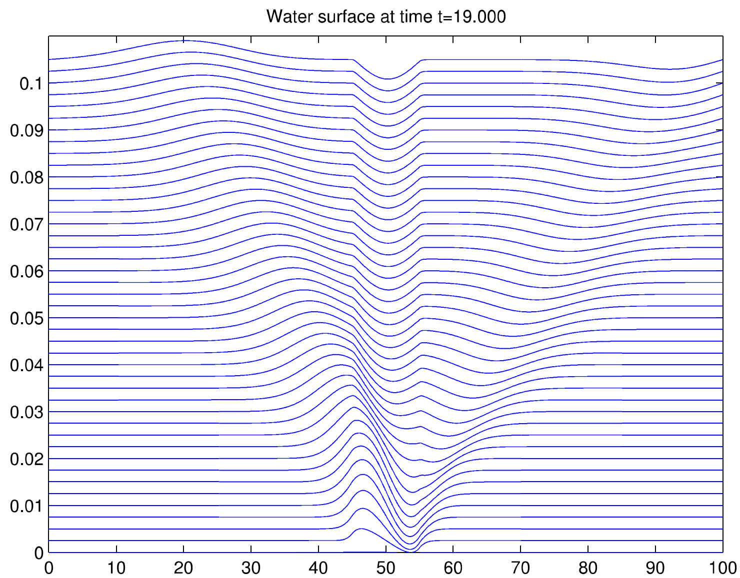

FIG. 3.1. *Waves generated by flow over a porous medium, where $F = 0.2$.*

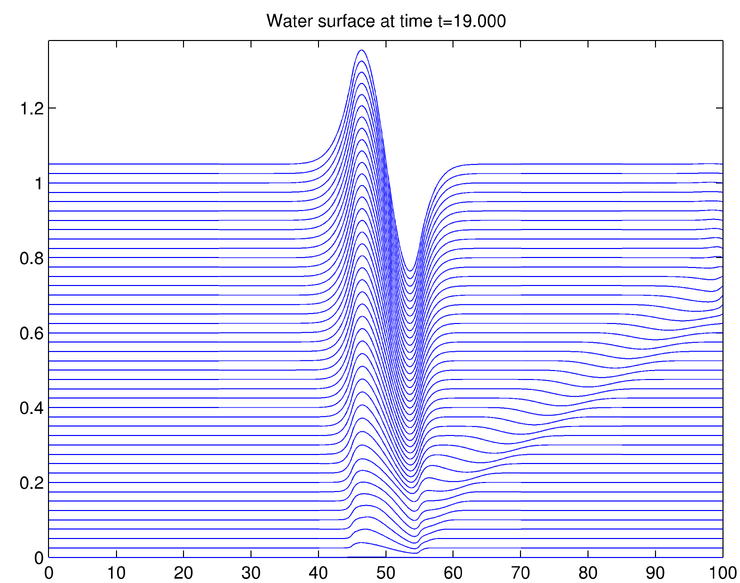

FIG. 3.2. *Waves generated by flow over a porous medium, where $F = 1.0$.*

For $F = 0.2$ the flow is subcritical. After the running time is sufficient, we see three waves. The first is unsteady and propagates to the left. The second is steady and keeps staying on the irregular part of the bed. The third is unsteady and moves to the right. At time $t = 19$ seconds, the results are shown in FIG. 3.1.

For $F = 1.0$ the flow is critical. We have two waves. One wave propagates to the right. The other one is steady on the irregular part of the bed. These are illustrated in FIG. 3.2.

For $F = 1.6$ and $F = 2.6$ the flow is supercritical. Three waves appear. Two of them are unsteady and both move to the right. The third wave is steady on the irregular part of the bed. What we can

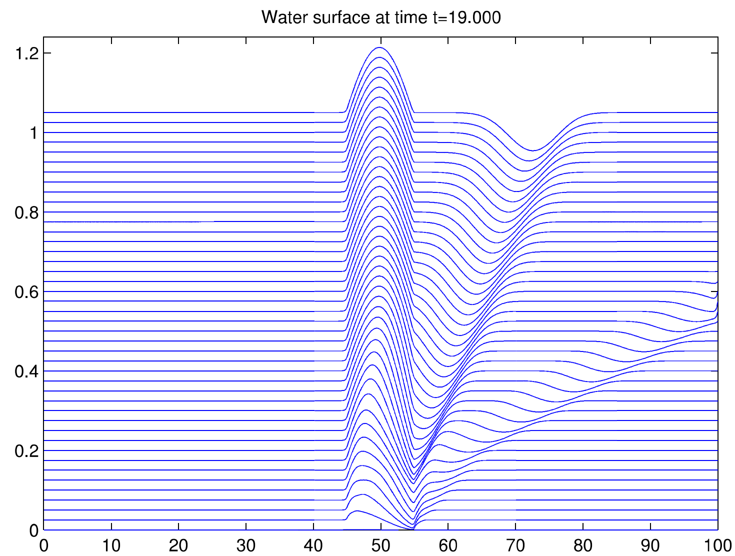

FIG. 3.3. *Waves generated by flow over a porous medium, where* $F = 1.6$.

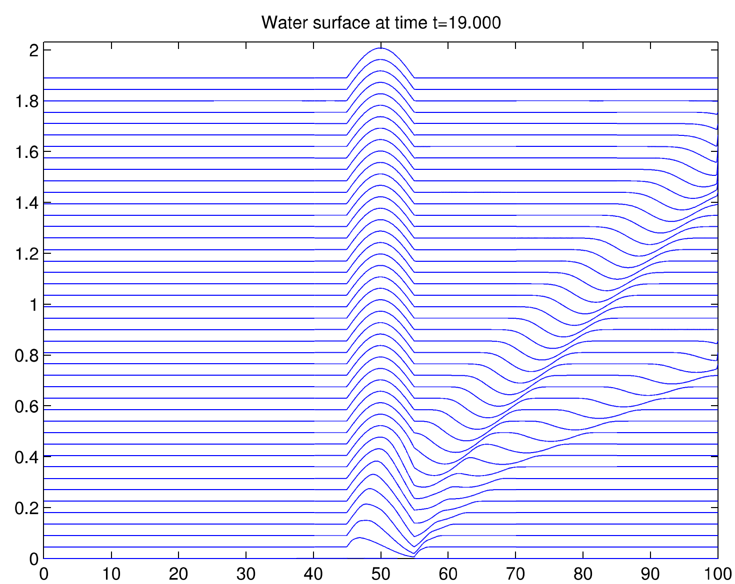

FIG. 3.4. *Waves generated by flow over a porous medium, where* $F = 2.6$.

infer from the results for $F = 1.6$ and $F = 2.6$ is that the greater Froude number leads to the faster propagation of the unsteady waves, which is physically correct. These are presented in FIG. 3.3 and FIG. 3.4.

The numerical results that we have in this work are consistent with those of Wiryanto [1]. The phenomena that we obtain using the promoted numerical method are the same as the phenomena in the previous work of Wiryanto [1]. However, note that our promoted method is explicit and does not involve any systems of linear equations. This makes the numerical method that we use here is simpler than the method used by Wiryanto [1].

4. Conclusions. We have shown the capability of the explicit predictor-explicit corrector method to solve unsteady wave problems generated by flow over porous media. The predictor step uses Adams-Bashforth iteration. The corrector step uses Adams-Moulton iteration. The method has third order of accuracy in both predictor and corrector steps. Using this method, steady state solutions can be obtained relatively fast. Unsteady waves can be obtained clearly. Furthermore, when a higher order method is desired, this method may be extended to achieve that goal by changing both predictor and corrector steps to higher order.

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