BUKTIKAN $\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$! (1)

BUKTI : RUMUS GAMMA

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx}$$

MISALKAN : $\boxed{x = y^2} \longrightarrow \frac{dx}{dy} = 2y$

$dx = 2y\,dy$

$$\Gamma(n) = \int_0^\infty e^{-y^2} (y^2)^{n-1} . (2y\,dy)$$

$$\Gamma(n) = 2\int_0^\infty e^{-y^2} \underbrace{y^{2n-2} . y}\, dy$$

$$\Gamma(n) = 2\int_0^\infty e^{-y^2} y^{2n-1}\, dy$$

$$\boxed{n = \tfrac{1}{2}} \longrightarrow \Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-y^2} y^{2\left(\frac{1}{2}\right)-1}\, dy$$

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-y^2} \underset{1}{y^0}\, dy$$

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-y^2}\, dy \qquad \times \tfrac{1}{2}$$

$$\frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-y^2}\, dy$$

ANALOG : $\frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-x^2}\, dx$

$$\longrightarrow \frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) \cdot \frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-y^2} dy \cdot \int_0^\infty e^{-x^2} dx$$

$$\frac{1}{4}\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \int_{x=0}^\infty \int_{y=0}^\infty e^{-x^2} e^{-y^2}\, dy . dx$$

$$\frac{1}{4}\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \int_{x=0}^\infty \int_{y=0}^\infty e^{-(x^2+y^2)} . \underbrace{dy.dx}_{dA}$$

$$\boxed{\begin{aligned} x^2 + y^2 &= r^2 \\ dA &= r\, dr\, d\theta \end{aligned}} , \quad \boxed{\begin{aligned} 0 \le \theta &\le \tfrac{\pi}{2} \\ 0 \le r &\le \infty \end{aligned}}$$

$$\frac{1}{4}\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{\infty} e^{-r^2} \cdot (r\,dr\,d\theta)$$

$$\frac{1}{4}\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \left[\int_{\theta=0}^{\frac{\pi}{2}} d\theta\right] \cdot \left[\int_{r=0}^{\infty} e^{-r^2}\, r\,dr\right]$$

$$= \left[\theta \Big|_0^{\frac{\pi}{2}}\right] \cdot \left[\frac{1}{-2}\int_{r=0}^{\infty} e^{-r^2} \cdot \overset{-2(r\,dr)}{d(-r^2)}\right]$$

$$= \left(\frac{\pi}{2} - 0\right) \cdot \left(-\frac{1}{2}\right) \cdot \lim_{p\to\infty} \int_{r=0}^{p} e^{-r^2}\, d(-r^2)$$

$$= -\frac{\pi}{4} \lim_{p\to\infty} \left[e^{-r^2}\Big|_0^p\right]$$

$$= -\frac{\pi}{4} \lim_{p\to\infty} \left[\frac{1}{e^{r^2}}\Big|_0^p\right]$$

$$= -\frac{\pi}{4} \lim_{p\to\infty} \left[\frac{1}{e^{p^2}} - \frac{1}{e^{0^2}}\right]$$

$$= -\frac{\pi}{4} \left[\lim_{p\to\infty} \frac{1}{e^{p^2}} - \lim_{p\to\infty} \frac{1}{e^{0}}\right]$$

$$= -\frac{\pi}{4} \left[\lim_{p\to\infty} \frac{1}{e^{\infty^2}} - \frac{1}{1}\right]$$

$$= -\frac{\pi}{4} \left[0 - 1\right]$$

$$\frac{1}{4}\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \frac{\pi}{4} \qquad \times 4$$

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = \pi$$

$$\boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$